Karlsruher Institut für Technologie

Completeness Theorems for k-SUM and Geometric Friends:
Deciding Integer Linear Arithmetic Fragments.

Geri Gokaj[1] and Marvin Künnemann[1]

[1]Karlsruhe Institute of Technology

ITCS' 25

The k-SUM problem

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How to formalize the notion of "captures"?
⇝ The notion of fine-grained completeness.

Building a class of problems

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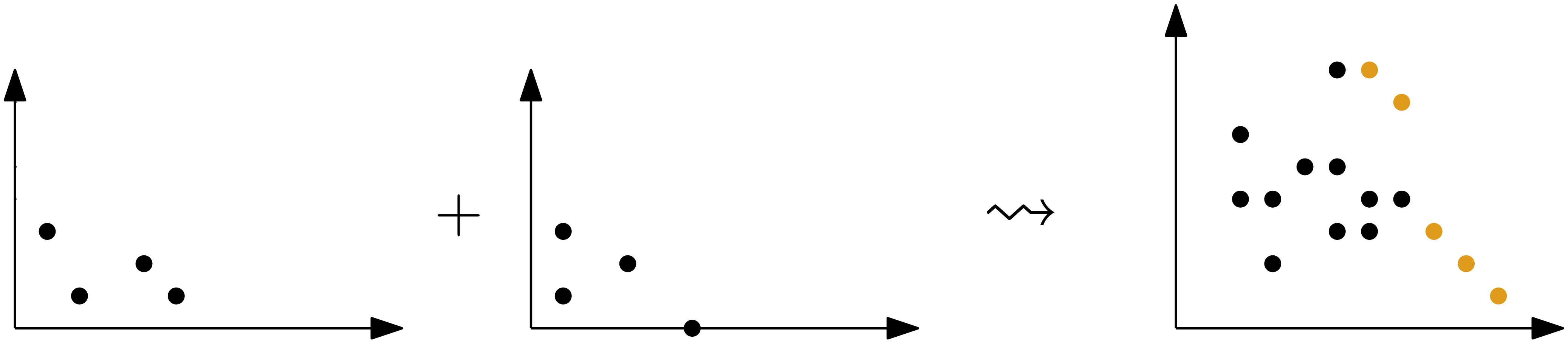

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$A + B := \{a + b : a \in A, b \in B\}$ black points.

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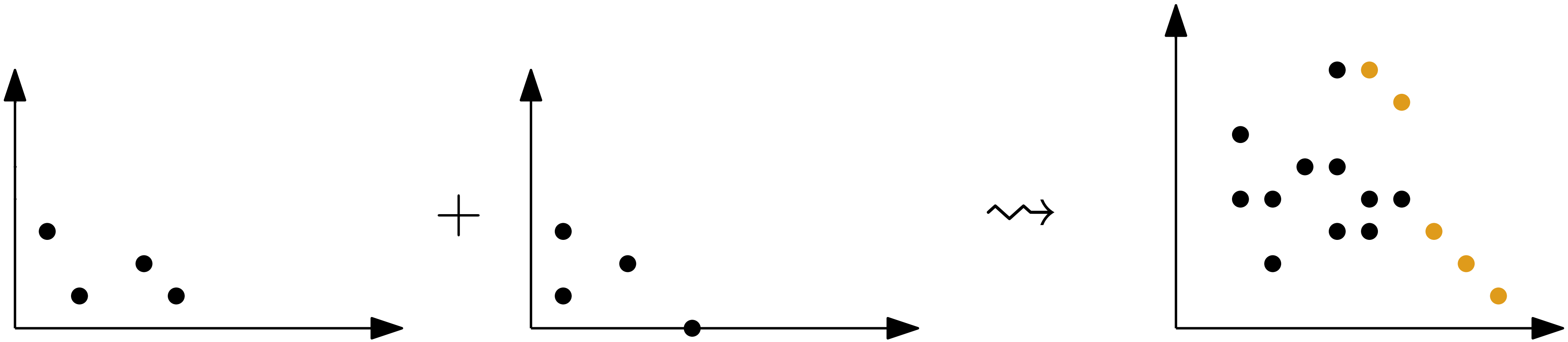

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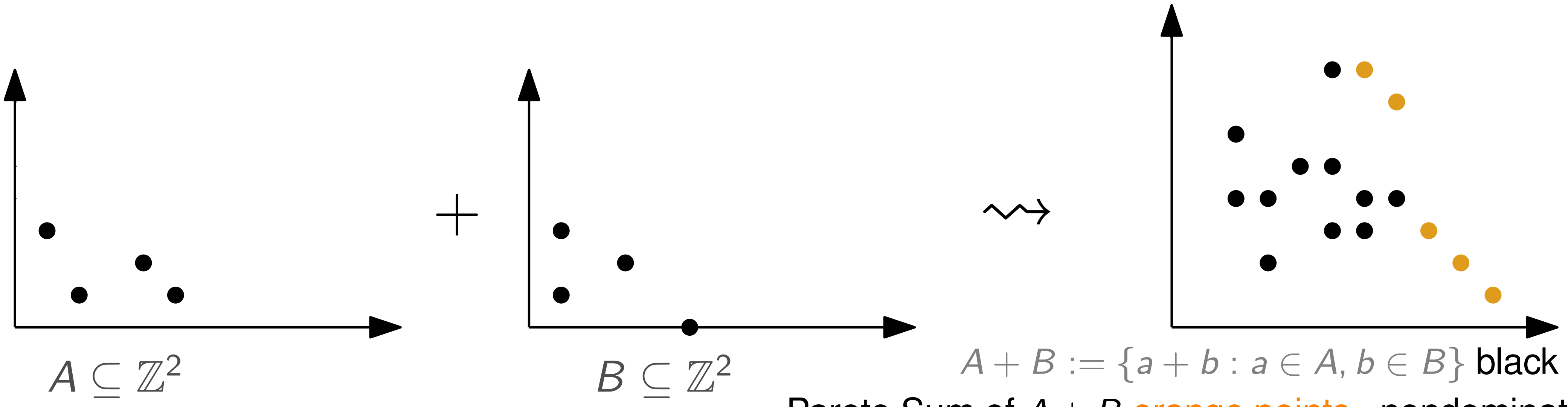

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Context of Approximation Algorithms [Wenk,02]

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Given: C in time $T_C(n)$ and a class of problems $\mathcal{C}$ in time $T_{\mathcal{C}}(n)$.

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Why fine-grained completeness?

- Algorithmic consequences,
- Class-based hardness.

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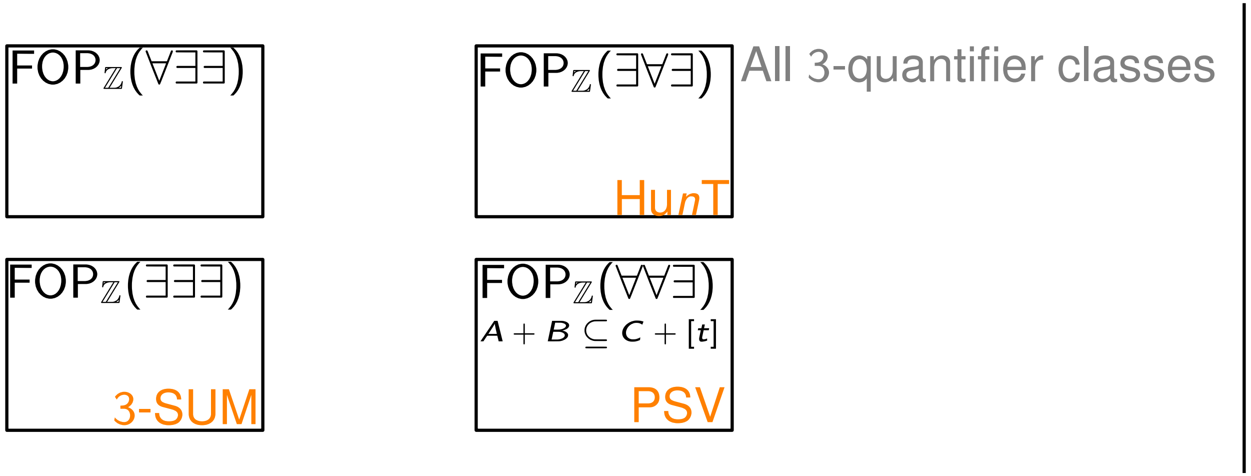

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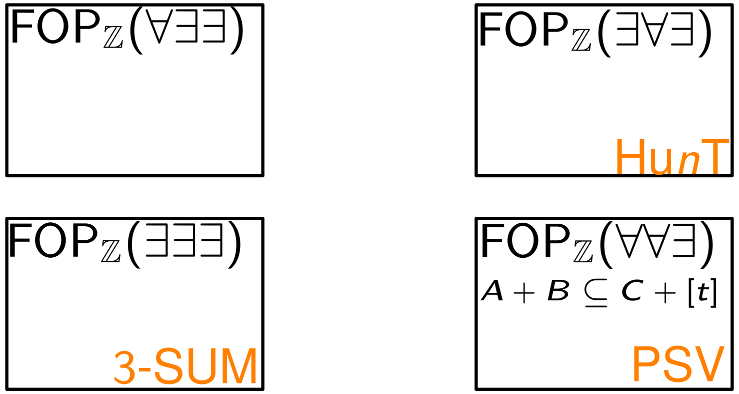

All 3-quantifier classes

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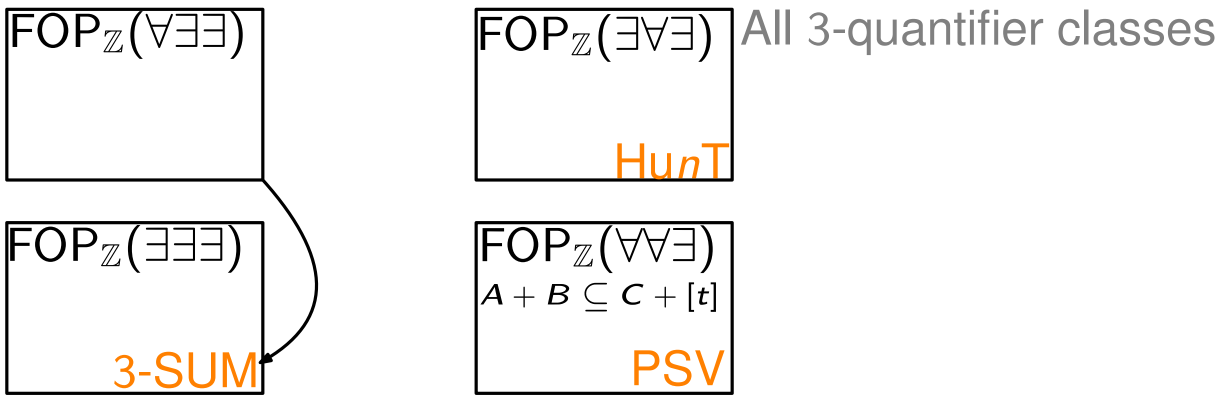

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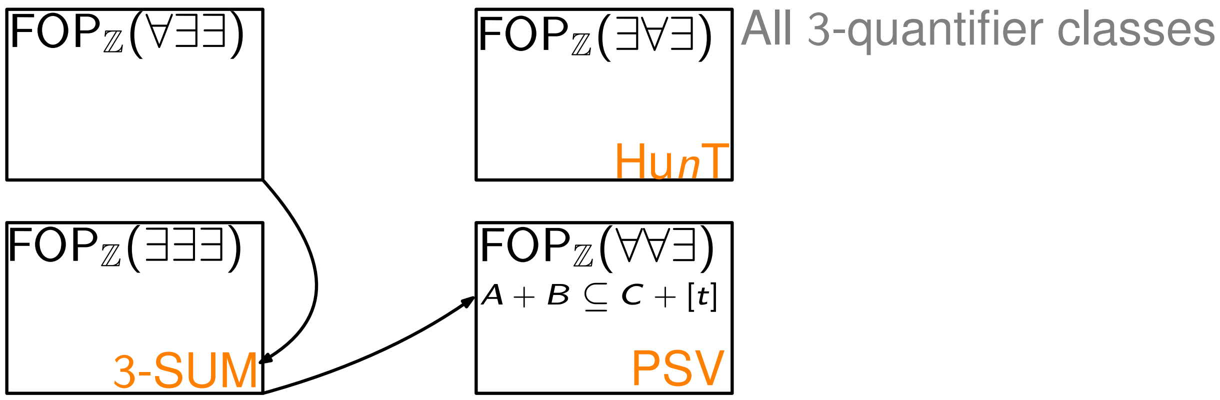

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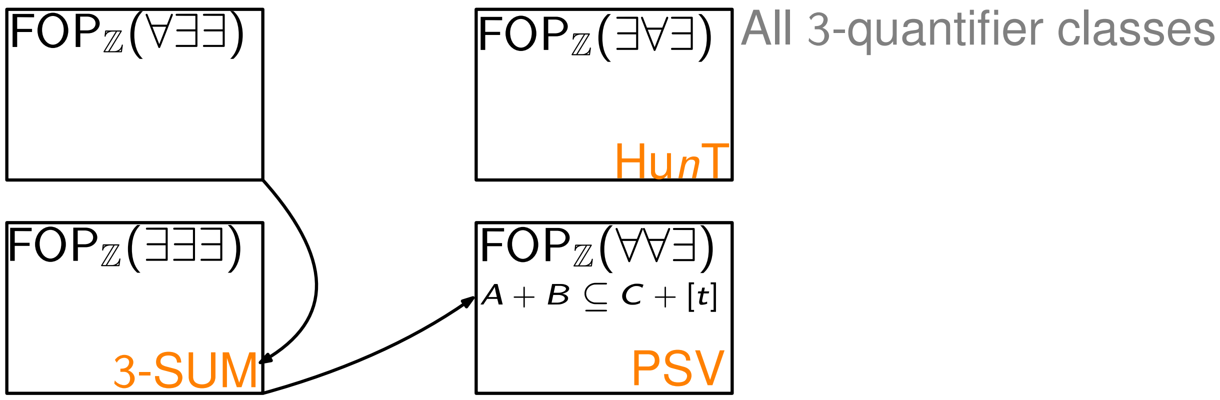

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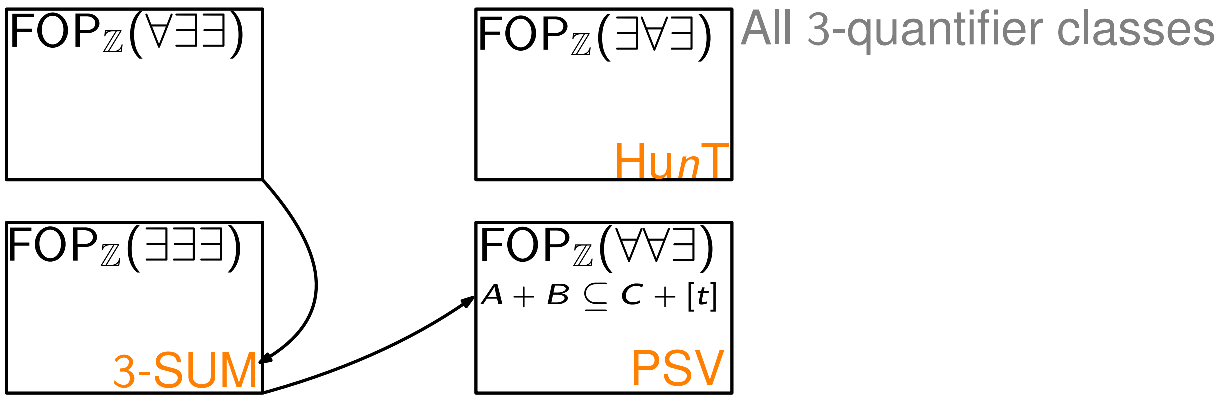

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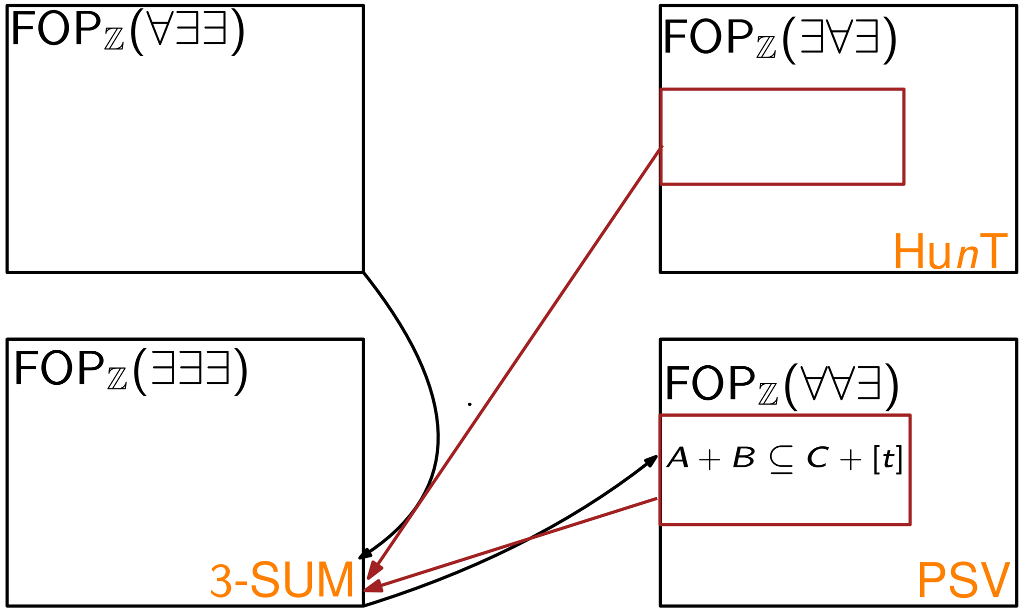

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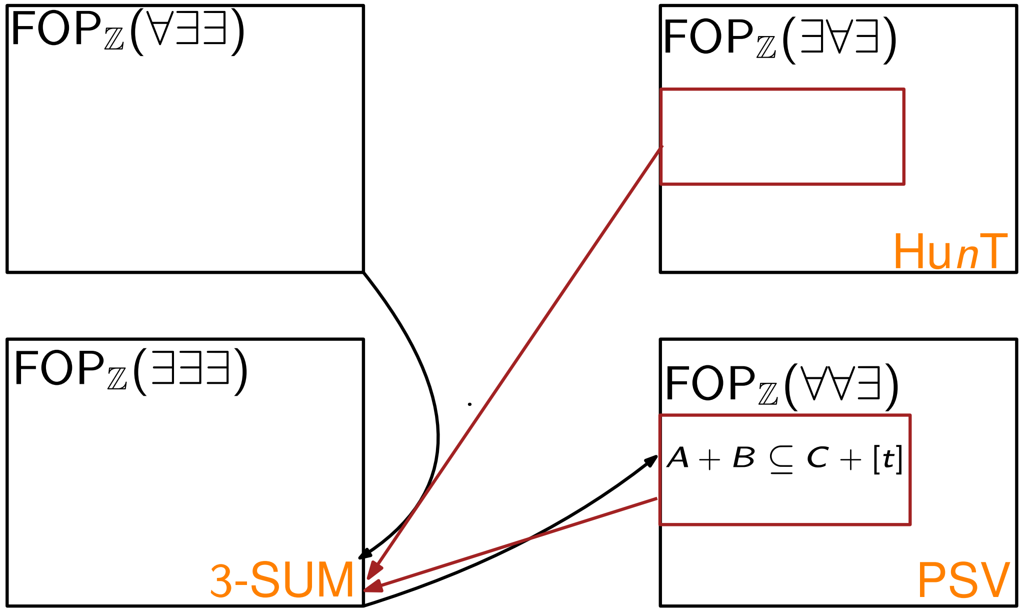
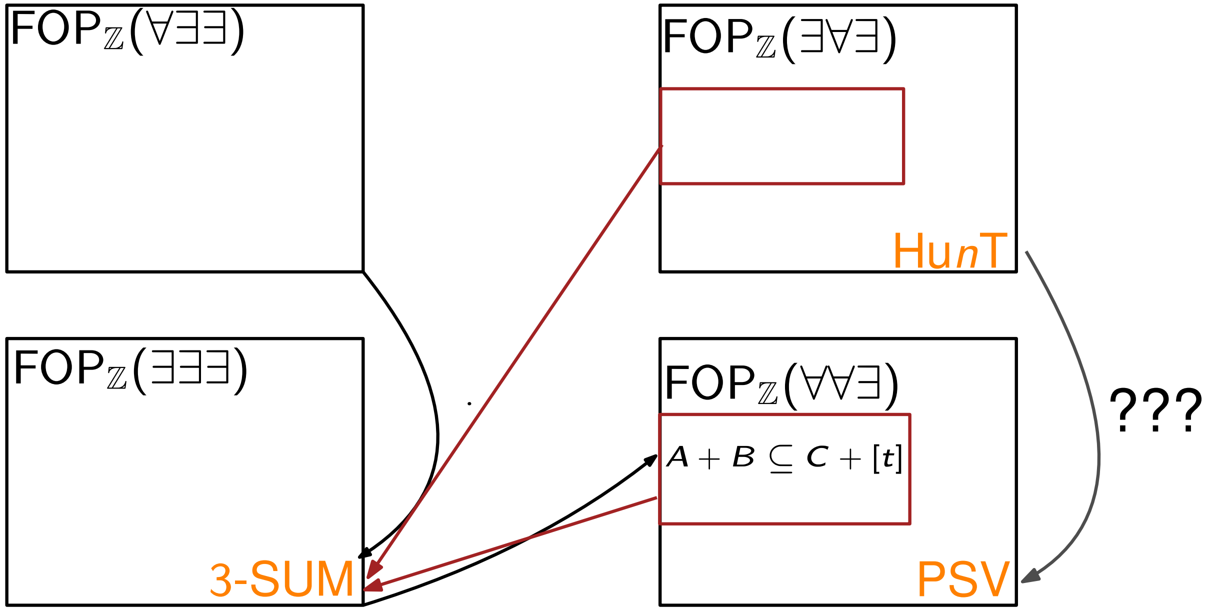
Definition: LIA formula φ has inequality dimension 3 if equivalent to a φ', with at most 3 atoms.

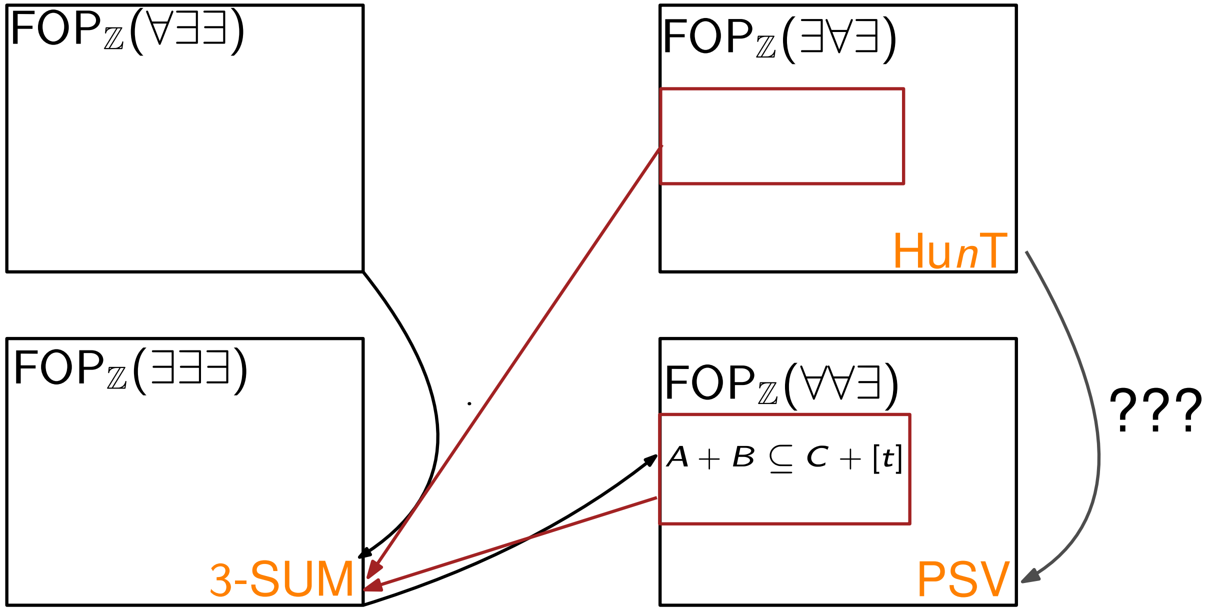


Proof counts witnesses to check quantifier structure. Crucially uses #3-SUM $\equiv_2$ 3−SUM [Chan, Willliams, Xu,23].

Big open question: HunT $\rightarrow$ PSV?



Big open question: HunT $\rightarrow$ PSV?



Thank you!